Heat Transfer with Ablation in a Finite Slab Subjected to Time-Variant Heat Fluxes

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Heat transfer with ablation in a finite slab subjected to time-variant heat fluxes is studied. Three different approaches are employed: the heat balance integral method, the θ-moment integal method, and an implicit finite difference method. Three specific heat fluxes are considered, in terms of linear, exponential, and "power law" functions of time. Numerical results for ablation distance and speed are presented based on the aforementioned techniques.

Nomenclature

c = heat capacity per unit volume
g = a time dependent parameter
H = heat of ablation
i = index for dimensionless space
j = index for dimensionless time
k = thermal conductivity
L = thickness of the slab
N = total number of elements
q_0 = dimensional wall heat flux
q_r = reference heat flux, $=k(T_m-T_0)/L$
Q = dimensionless heat flux, $=q_0L/k(T_m-T_0)$
S = ablation distance
s = dimensionless ablation distance, $=S/L$
$\dot{s}$ = $\mathrm{d}S/\mathrm{d}t$
t = dimensional time
t_m = time at the instant when melting starts
t_c = characteristic time, $=L^2/\alpha$
t_r = referenced time
T = dimensional temperature
T_0 = initial temperature
T_i = temperature of slab at the onset of melting
T_m = melting temperature
V = ablation speed, $=\mathrm{d}S/\mathrm{d}t$
X = coordinate along the slab
x = X/L
z = $x-s$
α = thermal diffusivity
β = a short time
η = a time dependent parameter
$\dot{\eta}$ = $\mathrm{d}\eta/\mathrm{d}\tau$
λ = the fraction of remaining element at the insulated boundary
τ = dimensionless time, $=t/t_c$
τ_m = dimensionless time when melting just starts, $=t_m/t_c$
τ_r = dimensionless reference time, $=t_r/t_c$
θ = $(T-T_0)/(T_m-T_0)$
θ_i = $(T_i-T_m)/(T_0-T_m)$
ν = inverse of Stefan number, $=H/c(T_m-T_0)$
ζ = $\tau-\tau_m$

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Introduction

HEAT transfer in solids involving ablation is of importance in numerous engineering applications. Examples include high speed re-entry of space vehicles, nuclear reactor operation, metal casting, to mention a few. Problems of this type are inherently nonlinear and involve a moving boundary which is not known a priori. The exact analytical solution for transient heat transfer in a solid accompanied by ablation is practically nonexistent. Only numerical and approximate analytical solutions have been made available.

Landau,[1] and Sunderland and Grosh,[2] by applying finite difference techniques, solved the ablation speed in a semi-infinite solid subject to a uniform heat flux or convection at the boundary. The Landau's problem was further attacked by Goodman[3,4] and Altman[5] using the heat balance integral method. Zien[6,7] later proposed the θ-moment integral method to treat one-dimensional transient ablation with time-dependent surface heat flux of the forms $q_0 \sim e^t$ and $q_0 \sim t^m$. Recently, Chung et al.[8] presented comparisons among the results of the classical heat balance integral method, Zien's θ-moment integral method, and a numerical finite difference approach as applied to the ablation problem with time-variant heat fluxes. For simplicity, all of the above references were restricted to the case where the ablator is semi-infinitely extant. In reality, the semi-infinite model tends to underpredict the melting rate because of the finite dimension of the actual structure. The purpose of the present study is to extend the analysis of Ref. 8 to a finite region.

Mathematical Analysis

Consider the case of one-dimensional heat conduction in a finite slab which is initially at a uniform temperature T_0. The solid is subjected to a time variant heat flux at the front surface ($X=0$), and insulated at the other ($X=L$). The governing equations for the preablation and ablation periods can be described, in dimensionless form, as follows:

Preablation Period

$$\frac{\partial \theta}{\partial \tau}=\frac{\partial^2 \theta}{\partial x^2} \qquad 0<\tau<\tau_m,\ 0<x<1 \tag{1}$$

$$\theta=0 \qquad \text{at } \tau=0,\ 0<x<1 \tag{2}$$

$$\frac{\partial \theta}{\partial x}=0 \qquad \text{at } x=1 \tag{3}$$

$$-\frac{\partial \theta}{\partial x}=Q \qquad \text{at } x=0 \tag{4}$$

Ablation Period

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial x^2} \qquad \tau_m<\tau,\ s<x<1 \tag{5}$$

$$\theta=\theta_i \qquad \text{at } \tau=\tau_m,\ 0<x<1 \tag{6}$$

$$\frac{\partial\theta}{\partial x}=0 \qquad \text{at } x=1 \tag{7}$$

$$\theta=1 \qquad \text{at } x=s \tag{8}$$

$$-\frac{\partial\theta}{\partial x}+\nu\frac{\mathrm{d}s}{\mathrm{d}\tau}=Q \qquad \text{at } x=s \tag{9}$$

where θ_i is the dimensionless temperature distribution at time τ_m. Both θ_i and τ_m are obtained from the preablation period. Equations (1-9) will be solved by using the three different approaches mentioned previously.

Heat Balance Integral Analysis

Preablation Period

Integrating Eq. (1) from 0 to 1 and employing Eqs. (3) and (4) and Leibnitz's rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1\theta\mathrm{d}x=Q \tag{10}$$

An exponential form of temperature profile is assumed

$$\theta=Q\eta\ \exp\left(-\frac{x}{\eta}\right) \tag{11}$$

where η is a time parameter to be determined. Note that the above equation satisfies the boundary condition at $X=0$, although it does not enforce the boundary condition given by Eq. (3). However, this condition is utilized in the derivation of Eq. (10). Substituting Eq. (11) into Eq. (10) yields

$$\dot{\eta}\left\{-2Q\eta\left[\exp\left(-\frac{1}{\eta}\right)-1\right]-Q\ \exp\left(-\frac{1}{\eta}\right)\right\}$$
$$=Q+\dot{Q}\eta^2\left[\exp\left(-\frac{1}{\eta}\right)-1\right] \tag{12}$$

where the overdot denotes a time derivative.

Equation (12) can be integrated numerically to yield η and, after substituting η into Eq. (11), gives the temperature distribution. The commencing time τ_m is determined by setting $\theta(0,\tau_m)=Q\eta=1$. Once τ_m is known, one can calculate $\theta_i=\theta(x,\tau_m)$ and proceed with the calculation for the ablation period.

It should be pointed out that the concept of postpenetration was generally employed for the case of heat conduction in a finite region.[9,10] Using this concept, the predicted temperature profiles are improved, but the computation procedures become much more involved if the phase change is taken into consideration. In a way, this approach defeats the purpose of the heat balance integral technique—simplicity. For this reason, the postpenetration concept in the current analysis is discarded. The idea of the present approach is to sacrifice the accuracy of the temperature profile somewhat in order to simplify the calculation steps considerably. It is believed that the addition of a new temperature profile for the post-penetration period to satisfy the boundary condition given by Eq. (3) has little effect on the prediction of melting rate and melting distance, which are the main desired items in this work. To demonstrate this point, the same problem will be solved using the θ-moment integral method and the finite difference method. As will be seen later, the agreement among these methods is quite acceptable.

Ablation Period

For the ablation period, one integrates Eq. (5) from the recession front $s(\tau)$ to 1, and makes use of Eqs. (7-9) along with Leibnitz's rule. This results in the integral equation for the ablation period:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_s^1\theta\mathrm{d}x+(1+\nu)\frac{\mathrm{d}s}{\mathrm{d}\tau}=Q \tag{13}$$

In light of the continuity of temperature between the preablation and ablation periods, the following temperature profile is assumed:

$$\theta=\exp\left[-\frac{x-s}{\eta(1-s)}\right] \tag{14}$$

This expression satisfies Eq. (8) during the entire ablation period. Equations (9) and (13) can now be rewritten as

$$\dot{s}=\left[Q-\frac{1}{\eta(1-s)}\right]\Big/\nu \tag{15}$$

and

$$\dot{\eta}\left\{-(1-s)\left[\exp\left(-\frac{1}{\eta}\right)-1\right]-\frac{(1-s)}{\eta}\exp\left(-\frac{1}{\eta}\right)\right\}$$
$$+\dot{s}\left\{\eta\left[\exp\left(-\frac{1}{\eta}\right)-1\right]+1+\nu\right\}=Q \tag{16}$$

Equations (15) and (16) are solved by using the standard fourth-order Runge-Kutta method. The initial conditions associated with this time period are $s=0$, $\eta=\eta_m$ at $\tau=\tau_m$. The ablation speed can be found in Eq. (15). This concludes the heat balance integral analysis.

θ-Moment Integral Analysis

Preablation Period

As is in heat balance integral method, Eq. (10) still holds. A second integral equation is derived by multiplying θ on both

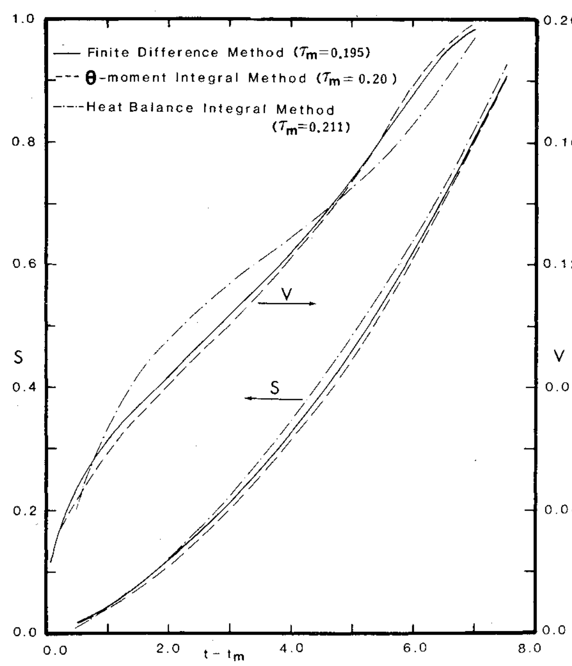

Fig. 1 Comparisons of ablation thickness and speed of various methods for $Q=2$.

sides of Eq. (1) and then integrating the result from 0 to 1, to yield

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 \theta^2 dx = Q\theta\bigg|_{x=0} - \int_0^1\left(\frac{\partial\theta}{\partial x}\right)^2 dx \tag{17}$$

where Eqs. (3) and (4) and Leibnitz's rule have been employed. Similar to Eq. (10), an exponential profile is assumed for θ with an additional time parameter, g.

$$\theta = g\eta\ \exp\left(-\frac{x}{\eta}\right) \tag{18}$$

By substituting Eq. (18) into Eqs. (10) and (17), one obtains, after rearrangement, the following system of ordinary differential equations:

$$\dot{g} = (B_3D_2 - B_2D_3)/(B_1D_2 - B_2D_1) \tag{19}$$

$$\dot{\eta} = (B_1D_3 - B_3D_1)/(B_1D_2 - B_2D_1) \tag{20}$$

where

$$B_1 = -\eta^2\left[\exp\left(-\frac{1}{\eta}\right) - 1\right]$$

$$B_2 = -2g\eta\left[\exp\left(-\frac{1}{\eta}\right) - 1\right] - g\ \exp\left(-\frac{1}{\eta}\right)$$

$$B_3 = Q$$

$$D_1 = -\frac{1}{2}\eta^2\left[\exp\left(-\frac{2}{\eta}\right) - 1\right]$$

$$D_2 = -\frac{3}{4}g\eta\left[\exp\left(-\frac{2}{\eta}\right) - 1\right] - \frac{1}{2}g\ \exp\left(-\frac{2}{\eta}\right)$$

$$D_3 = Q + \frac{1}{2}g\left[\exp\left(-\frac{2}{\eta}\right) - 1\right]$$

Note that the denominator of the right-hand side of Eqs. (19) and (20) vanishes at $\tau = 0$. To avoid this singularity, a short time solution at $x = 0$ is obtained by using Laplace transform as follows:

$$\theta(0,\beta) = \frac{1}{\sqrt{\pi}}\int_0^\beta \frac{1}{\sqrt{t'}}Q(\beta - t')dt' \tag{21}$$

where β is an arbitrary small time. The short time conditions for g and η can be determined by using Eqs. (4) and (21)

$$g(\beta) = Q \tag{22}$$

and

$$\eta(\beta) = (1/Q)\theta(0,\beta) \tag{23}$$

where $\theta(0,\beta)$ is obtained from Eq. (21). With Eqs. (22) and (23) as initial conditions, Eqs. (19) and (20) can be easily integrated to give g and η as functions of time.

Ablation Period

For this time period, the corresponding integral equations are Eqs. (13) and (24) given below

$$\frac{1}{2}\frac{d}{d\tau}\int_s^1 \theta^2 dx + \left(\frac{1}{2} + \nu\right)\frac{ds}{d\tau} = Q - \int_s^1\left(\frac{\partial\theta}{\partial x}\right)^2 dx \tag{24}$$

which is derived in the same manner as that of Eq. (17). The following expression for the temperature profile at ablation period is assumed

$$\theta = \exp\left[-\frac{x-s}{\eta(1-s)}\right] \tag{25}$$

By substituting Eq. (25) into Eqs. (13) and (24), one arrives at

$$\dot{s} = (P_3R_2 - P_2R_3)/(P_1R_2 - P_2R_1) \tag{26}$$

$$\dot{\eta} = (P_1R_3 - P_3R_1)/(P_1R_2 - P_2R_1) \tag{27}$$

where

$$P_1 = \eta\left[\exp\left(-\frac{1}{\eta}\right) - 1\right] + 1 + \nu$$

$$P_2 = -(1-s)\left\{\exp\left(-\frac{1}{\eta}\right) - 1 + \frac{1}{\eta}\exp\left(-\frac{1}{\eta}\right)\right\}$$

$$P_3 = Q$$

$$R_1 = \frac{\eta}{4}\left[\exp\left(-\frac{2}{\eta}\right) - 1\right] + \frac{1}{2} + \nu$$

$$R_2 = -(1-s)\left\{\frac{1}{4}\left[\exp\left(-\frac{2}{\eta}\right) - 1\right] + \frac{1}{2\eta}\exp\left(-\frac{2}{\eta}\right)\right\}$$

$$R_3 = Q + \frac{1}{2\eta(1-s)}\left[\exp\left(-\frac{2}{\eta}\right) - 1\right]$$

With initial conditions $s(\tau_m) = 0$ and $\eta(\tau_m)$ obtained from preablation period at $\tau = \tau_m$, Eqs. (26) and (27) can be integrated to yield s and η. The ablation speed $\dot{s}$ is then solved from Eq. (26).

Finite Different Analysis

Preablation Period

With boundary heat flux Q specified, Eqs. (1-4) can be solved exactly using Laplace transform and convolution theorem. The analytical solutions are given below for various types of heat fluxes.

$$\theta(x,\tau) = Q_0\left\{\tau + \frac{1}{2}x^2 - \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2}\times\exp(-n^2\pi^2\tau)\cos(n\pi x)\right\} \tag{28}$$

for $Q = Q_0 = \text{const}$

$$\theta(x,\tau) = Q_0\left\{\frac{1}{2}\tau^2 - \frac{1}{6}\tau + \frac{1}{2}x^2\tau + \frac{7}{360} - \frac{1}{12}x^2 + \frac{1}{24}x^4 + \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\exp(-n^2\pi^2\tau)\cos(n\pi x)\right\} \tag{29}$$

for $Q = Q_0\tau$,

$$\theta(x,\tau) = \frac{1}{3}Q_0\left\{\tau + \frac{3}{2}x^2\tau^2 - \frac{1}{2}\tau^2 + \frac{1}{4}\tau x^4 - \frac{1}{2}x^2\tau + \frac{7}{60}\tau + \frac{1}{120}x^6 - \frac{1}{24}x^4 + \frac{7}{120}x^2 - \frac{31}{2520} - 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^6\pi^6}\exp(-n^2\pi^2\tau)\cos(n\pi x)\right\} \tag{30}$$

for $Q = Q_0 \tau^2$, and

$$\theta(x,\tau) = Q_0 \tau_r \left\{ -1 + \frac{1}{\tau_r} \frac{\exp(\tau/\tau_r)\cos(x/\tau_r)}{\sinh(1/\tau_r)} \right\} \tag{31}$$

for $Q = Q_0 \exp(\tau/\tau_r)$, where $\tau_r = t_r / t_c$, t_r being a reference time. The commencing time τ_m is solved from these equations by setting $\theta(0,\tau_m) = 1$.

Ablation Period

To simplify the analysis, the following coordinate transformation is adopted first:

$$z = x - s \tag{32}$$

$$\xi = \tau - \tau_m \tag{33}$$

Equations (5-9) are reduced to

$$\frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial z^2} - \dot{s}\frac{\partial \theta}{\partial z} \qquad 0<\xi,\ 0<z<1-s \tag{34}$$

$$\theta = \theta_i \qquad \text{at } \xi = 0\ \ 0<z<1 \tag{35}$$

$$\frac{\partial \theta}{\partial z} = 0 \qquad \text{at } z = 1 - s \tag{36}$$

$$\theta = 1 \qquad \text{at } z = 0 \tag{37}$$

$$-\frac{\partial \theta}{\partial z} + \nu \dot{s} = Q \qquad \text{at } z = 0 \tag{38}$$

Note that the use of the above coordinate transformations has immobilized the intricate ablating surface while mobilizing the insulated surface and, hence, simplifying the numerical scheme considerably.

An implicit finite difference scheme is employed here to deal with Eqs. (34-38). One approximates the time derivative by forward difference and space derivative by central difference, and Eq. (34) becomes

$$\left(\frac{1}{\Delta z^2} + \frac{\dot{s}}{2\Delta z}\right)\theta_{i-1,j+1} + \left(-\frac{2}{\Delta z^2} - \frac{1}{\Delta \xi}\right)\theta_{i,j+1} + \left(\frac{1}{\Delta z^2} - \frac{\dot{s}}{2\Delta z}\right)\theta_{i+1,j+1} = -\frac{1}{\Delta \xi}\theta_{i,j} \tag{39}$$

where $\theta(z,\xi) = \theta(i\Delta z, j\Delta\xi)$.

Once there is material loss, the element next to the insulated surface, which is currently moving, starts to decrease. A linear interpolation scheme is used to approximate the finite difference expression. Therefore, the finite difference representation of Eq. (34) for this end element reads

$$\frac{z}{(\lambda\Delta z)^2}\theta_{N-1,j+1} + \left[-\frac{2}{(\lambda\Delta z)^2} - \frac{1}{\Delta\xi}\right]\theta_{N,j+1} = -\frac{\theta_{N,j}}{\Delta\xi} \tag{40}$$

where λ is a fraction of the remaining element at the insulated side, and N is the total number of elements. Equations (39) and (40) with Eq. (41) below can be assembled in a matrix form and are resolved for the unknown nodal temperatures. The ablation speed is obtained from Eq. (38), by using a third-order approximation for $\partial\theta/\partial z$ at $z = 0$, as follows:

$$\dot{s} = \frac{1}{\nu}\left[Q + \frac{1}{2\Delta z}\left(-\frac{11}{3}\theta_{1,j} + 6\theta_{2,j} - 3\theta_{3,j} + \frac{2}{3}\theta_{4,j}\right)\right] \tag{41}$$

The corresponding ablation thickness is then calculated by integrating $\dot{s}$.

Once the ablation thickness exceeds Δz, the last element (insulated boundary element) is dropped and the rank of matrix is reduced by one. The above calculation procedures continue until the material is completely ablated.

Results and Discussion

Numerical results for ablation thickness and speed based on the aforementioned three methods are illustrated in Figs. 1-4. Four specific heat flux boundary conditions are chosen in the present computations, namely, $Q=2$, $Q=10\tau$, $Q=10\tau^2$, and $Q=0.1\ \exp(\tau/\tau_r)$. The thermal diffusivity, characteristic time, characteristic length, reference heat flux, temperature difference, the inverse of Stefan number, and the reference time employed in the figures are $\alpha = 0.1$ ft²/s, $t_c = 10$ s, $L = 1$

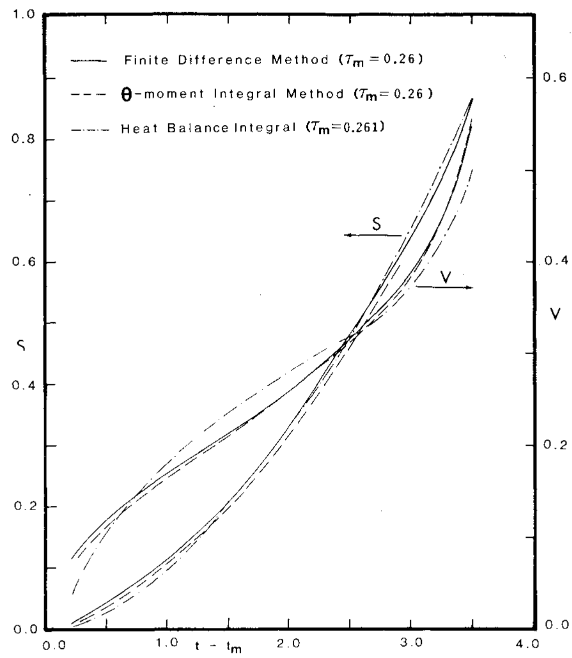

Fig. 2 Comparisons of ablation thickness and speed of various methods for $Q = 10\tau$.

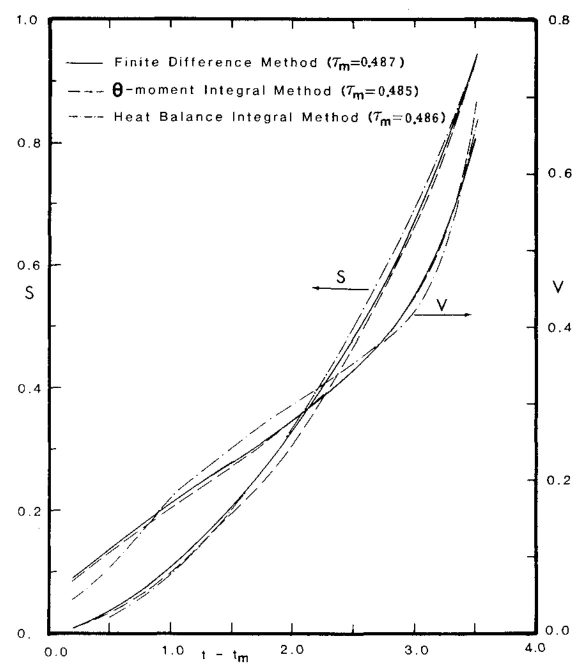

Fig. 3 Comparison of ablation thickness and speed of various methods for $Q = \tau^2$.

ft, $q_r = 10$ Btu/s-ft^2, $T_m - T_0 = 100$°F, $\nu = 1$, and $t_r = 100$ s, respectively. Note that the choice of the numerical values is arbitrary. They are for demonstration purposes only. For reentry problems, the assumption of uniform heat flux boundary condition appears to be oversimplified. Altman[5] has presented a typical time-dependent heat flux function; the peak of the heat flux curve might range, depending on condition of re-entry, from 100 to 10,000 Btu/s-ft^2 and the duration of re-entry ranges from 50 to 100 s. Zien[5] has pointed out that the heat flux in terms of $q_0 \sim t^m$ and $q_0 \sim e^t$ simulate more realistically the ablation phenomenon in the atmospheric entry environments than the uniform boundary heat flux model. A case with a power-law heat flux boundary condition has also been studied by Vallerani[11] for a semi-infinite solid.

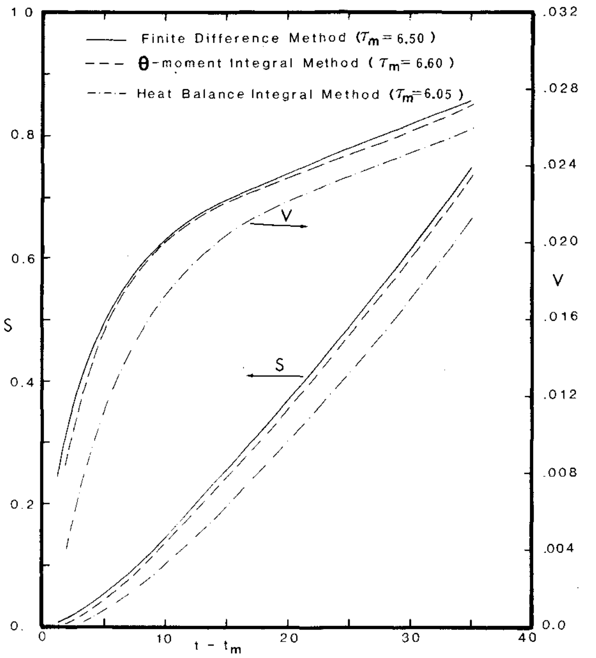

Fig. 4 Comparison of ablation thickness and speed of various methods for $Q = 0.1 \exp(\tau/\tau_r)$.

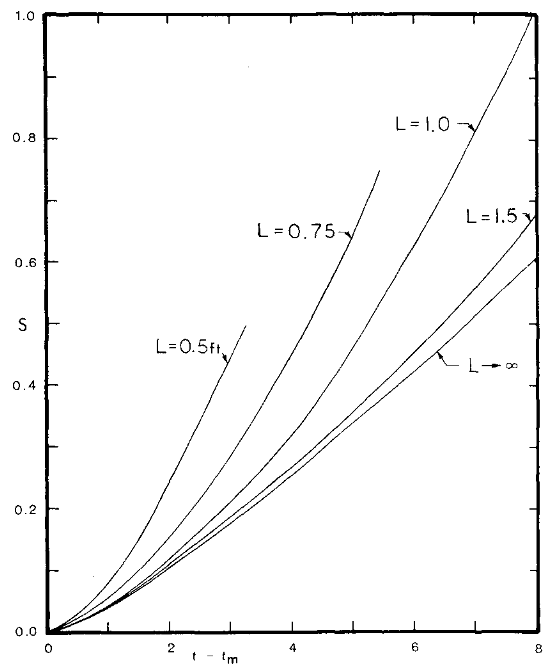

Fig. 5 Comparison of ablation thickness for various dimensions of slab with $q_0 = 20$ Btu/s-ft^2.

In the finite difference scheme, the mesh size used is continuously refined until there is no significant change in the solution. In the present output, 20 increments in space were employed. Furthermore, the numerical value of ablation commencing time t_m for this case is also computed from the exact analytical expression given by Eqs. (28-31). Therefore, the solid lines labeled under "Finite Difference Method" in Figs. 1-4 can be considered as exact. Examination of Figs. 1-3 reveals that the ablation thickness obtained from the θ-moment integral method yields slightly better results than that of heat balance integral method. However, Fig. 4 shows that the numerical solution of ablation thickness agrees well with that of θ-moment integral mehod but deviates somewhat from the heat balance integral solution.

The predicted ablation speeds based on the above three methods are also presented in Figs. 1-4 on the right-hand-side scale. In general, the θ-moment integral method shows a great improvement over the heat balance integral method. This is especially true for the case with exponential heat flux boundary condition. Note that all curves are plotted against dimensional time difference, $t - t_m$, instead of an absolute time scale. The values of τ_m for each case are also listed on the figures for reference. It is seen that the predicted τ_m based on all these methods agree very well. This further demonstrates the validity of the present simplified assumption of using Eq. (11) as a sole temperature profile during the preablation period. In fact, the accuracy of the heat balance integral method is not very sensitive to the choice of temperature profile for the preablation period. Özisik[12] has illustrated this point by considering heat conduction in a semi-infinite solid with a uniform initial temperature T_i and a step change in surface temperature, T_0. Using the heat balance integral method and choosing three different temperature profiles, he obtained the surface heat flux of the form $q_0 = c(T_0 - T_i)/\sqrt{\alpha t}$, where $c = 0.577$, 0.530, and 0.548 for the second, third, and fourth order polynomial of temperature profiles, respectively, while the exact solution for c is 0.565.

As pointed out earlier, Refs. 1-8 are restricted to the case of ablation in a semi-infinite region. The use of a semi-infinite model may not represent the physical reality accurately because of the finite dimension of the actual structure. Figure 5 demonstrates the effect of slab dimension L upon the ablation thickness. The results are based on the finite difference method and the heat flux at the boundary is uniformly distributed with $q_0 = 20$ Btu/s-ft^2. It is seen that the ablation speed for $L = 0.5$ ft is much faster than that of a semi-infinite solid. Furthermore, the total melting time is not linearly proportional to the total length of the slab. As expected intuitively, the use of a semi-infinite slab assumption in the analysis may appreciably underpredict the melting rate.

Due to the approximate nature of the heat balance integral and the θ-moment integral methods, their application is emphasized on the determination of ablation thickness and ablation rate instead of detailed temperature distribution inside the solid.

Acknowledgment

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